

Explicit Pre A^* -algebra**Habtu Alemayehu^{1*}, Venkateswararao, J¹ and Satyanarayana, A²**¹Department of Mathematics, CNCS, Mekelle University, P.B.No.231, Mekelle, Ethiopia (*habtua@yahoo.com).²Department of Mathematics, A.N.R College, Gudiwada, A.P. India.**ABSTRACT**

This manuscript is a study on Birkhoff centre of a Pre- A^* -algebra. In fact, it is proved that Birkhoff centre of a Pre A^* -algebra is also a Pre A^* -algebra and identified that the centre of Birkhoff centre of a Pre A^* -algebra is a Boolean algebra.

Keywords: Pre A^* -algebra, Centre, Birkhoff centre, Boolean algebra.**AMS Mathematics subject classification (2000):** 03G25(03G05, 08G05).**1. INTRODUCTION**

The notions of lattice concerned aspects were detailed conferred by Birkhoff (1948). In an outline script Manes (1993) initiated the perception of Ada, based on C-algebras by Fernando and Craig (1990).

Chandrasekhararao et al. (2007) bring in the impression Pre A^* -algebra $(A, \wedge, \vee, (-)^\sim)$ akin to C-algebra as a reduct of A^* - algebra. Venkateswararao et al. (2009) added the structural compatibility of Pre A^* -Algebra with Boolean algebra. Further, Satyanarayana and Venkateswararao (2011) ascertained the thought of ideals of Pre A^* -algebras. Boolean algebra depends on two element logic. C-algebra, Ada, A^* - algebra and our Pre A^* -algebra are standard expansions of Boolean logic to 3 truth values, anywhere the third truth value indicates an undefined one.

We recognize the Birkhoff Centre of a Pre A^* -algebra and attest various associated results as well. Swamy and Murti (1981) initiated the perception of centre of a C-algebra and bear out that it is a Boolean algebra through induced operations. Furthermore, Swamy and Pragathi (2003) initiated the observation of Birkhoff's centre of a semigroup and extended the above concept for a general semigroup and proved that the Birkhoff's centre of any semigroup is a relatively complemented distributive lattice.

Let us summon up with the objective of, if S is a semigroup and there exists $0, 1$ such that $x \cdot 0 = 0$, $x \cdot 1 = x$ and $1 \cdot x = x$ for all x belongs to S , then S is named a semigroup with 1 . An element a in S is referred as a Birkhoff central element of S if there exist semigroups say S_1 and S_2 with 0 and 1 an isomorphism S onto $S_1 \times S_2$ which maps a onto $(0, 1)$. The set with Birkhoff central elements of S is referred to be Birkhoff centre of S . This conception is extended to a Pre A^* -algebra with 1 and attested that the set of all central elements of a Pre A^* -algebra with 1 is a Pre A^* -algebra.

1. Preliminaries

1.1. Definition (Chandrasekhararao et al., 2007):

An algebra $(A, \wedge, \vee, (-)^\sim)$ where A is a non-empty set with 1 ; $\wedge, \vee$ are binary operations and

$(-)^\sim$ is a unary operation on A satisfying:

- (a) $x^{\sim\sim} = x$ for all x in A
- (b) $x \wedge x = x$ for all x in A
- (c) $x \wedge y = y \wedge x$ for all x in A
- (d) $(x \wedge y)^\sim = x^\sim \vee y^\sim$ for all x, y in A
- (e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ for all x, y, z in A
- (f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x, y, z in A
- (g) $x \wedge y = x \wedge (x^\sim \vee y)$ for all x, y in A is called a Pre A^* -algebra.

1.1. Example (Chandrasekhararao et al., 2007):

The set $\mathbf{3} = \{0, 1, 2\}$ by means of operations $\wedge, \vee, (-)^\sim$ defined below is a Pre A^* -algebra.

$\wedge$	0	1	2	$\vee$	0	1	2	x	$x^\sim$
0	0	0	2	0	0	1	2	0	1
1	0	1	2	1	1	1	2	1	0
2	2	2	2	2	2	2	2	2	2

1.1. Note (Chandrasekhararao et al., 2007): The above example make sense the following:

- (a) $2^\sim = 2$. (The only element in the set $\mathbf{3}$ with this property)
- (b) $1 \wedge x = x$ for all $x \in \mathbf{3}$. (1 is the meet ($\wedge$) identity in $\mathbf{3}$).
- (c) $0 \vee x = x$ for all $x \in \mathbf{3}$. (0 is the join ($\vee$) identity in $\mathbf{3}$).
- (d) $2 \wedge x = 2 \vee x = 2$ for all $x \in \mathbf{3}$.

1.2. Example (Chandrasekhararao et al., 2007):

$2 = \{0, 1\}$ with operations $\wedge, \vee, (-)^\sim$ defined below is a Pre A^* -algebra.

$\wedge$	0	1	$\vee$	0	1	x	$x^\sim$
0	0	0	0	0	1	0	1
1	0	1	1	1	1	1	0

1.2. Definition (Satyanarayana and Venkateswararao, 2011):

Let A be a Pre A^* -algebra. An element $x \in A$ is described as a central element of A if $x \vee x^\sim$ and the set $\{x \in A / x \vee x^\sim = 1\}$ of all central elements of A is referred the centre of A , denoted $B(A)$.

1.1. Theorem (Satyanarayana and Venkateswararao, 2011):

Let A be a Pre A^* -algebra with 1. Subsequently, $B(A)$ is a Boolean algebra in the midst of the operations $\wedge, \vee, (-)^\sim$.

1.1. Lemma (Satyanarayana and Venkateswararao, 2011):

Every Pre A^* -algebra with 1 satisfies the following:

- (a) $x \vee 1 = x \vee x^\sim$. (b) $x \wedge 0 = x \wedge x^\sim$.

1.2. Lemma (Satyanarayana and Venkateswararao, 2011):

Every Pre A^* -algebra by means of 1 satisfies the following:

- (a) $x \wedge (x^\sim \vee x) = x \vee (x^\sim \wedge x) = x$ (b) $(x \vee x^\sim) \wedge y = (x \wedge y) \vee (x^\sim \wedge y)$
(c) $(x \vee y) \wedge z = (x \wedge z) \vee (x^\sim \wedge y \wedge z)$

1.3. Definition (Satyanarayana and Venkateswararao, 2011):

Let A be a Pre A^* -algebra. An element x in A is a central element of A if $x \vee x^\sim = 1$ and the set $\{x \in A / x \vee x^\sim = 1\}$ of all central elements of A is referred the centre of A denoted $B(A)$.

1.4. Note (Venkateswararao et al., 2009): If A is a Pre A^* - algebra with 1, then 1, 0 are in $B(A)$.

If the centre of a Pre A^* - algebra concurs with $\{0, 1\}$, then we declare that A has trivial centre.

1.2. Theorem (Venkateswararao et al., 2009):

Let A be a Pre A^* -algebra with 1. Then $B(A)$ is a Boolean algebra by means of the operations $\wedge$, $\vee$, $(-)^{\sim}$.

2. BIRKHOFF'S CENTRE

In this segment, we describe Birkhoff centre of a Pre A^* - algebra, in addition we shall bear out assorted properties.

2.1. Definition:

Let A be a Pre A^* - algebra with meet identity. An element $a \in A$ is assumed to be Birkhoff central element of a Pre A^* - algebra A if there exist Pre A^* - algebras A_1 and A_2 with 1 (meet ($\wedge$) identity) and an isomorphism $f: A \rightarrow A_1 \times A_2$ such that $f(a) = (1_1, 0_2)$. (Where 1_1 , is the meet identity in A_1 and 0_2 is the join ($\vee$) identity in A_2 in that order).

2.2. Definition:

The set of all Birkhoff central elements of a Pre A^* - algebra A is described Birkhoff centre of A and denoted $BC(A)$.

2.1. Lemma:

Let A be a Pre A^* - algebra with meet identity. Then for each element $a \in BC(A)$ entails $a^{\sim} \in BC(A)$.

Proof: Let a be an element in $BC(A)$. Then there subsist Pre A^* - algebras A_1 and A_2 ; and an isomorphism $f: A \rightarrow A_1 \times A_2$ such that $f(a) = (1_1, 0_2)$.

Now, define $g: A \rightarrow A_2 \times A_1$ such that $g(x) = (x_2, x_1)$ whenever $f(x) = (x_1, x_2)$.

Let $x, y \in A$ such that $f(x) = (x_1, x_2)$ and $f(y) = (y_1, y_2)$.

At that moment, $f(x \wedge y) = (x_1 \wedge y_1, x_2 \wedge y_2)$, as f is a homomorphism.

Consequently we have, $g(x \wedge y) = (x_2 \wedge y_2, x_1 \wedge y_1)$

$$= (x_2, x_1) \wedge (y_2, y_1)$$

$$= g(x) \wedge g(y).$$

In the same way, we can provide evidence that, $g(x \vee y) = (x) \vee g(y)$.

To substantiate that $g(x^\sim) = [g(x)]^\sim$.

Let $f(x) = (x_1, x_2)$. Then $g(x) = (x_2, x_1)$.

Regard as, $f(x) = (x_1, x_2)$. This entails, $(f(x))^\sim = (x_1, x_2)^\sim = (x_1^\sim, x_2^\sim)$.

Hence we have, $f(x^\sim) = (x_1^\sim, x_2^\sim)$. (Since f is a homomorphism).

As a result, $g(x^\sim) = (x_2^\sim, x_1^\sim) = (x_2, x_1)^\sim = (g(x))^\sim$. Therefore, g is a homomorphism.

Besides reflect on, $f(a^\sim) = (f(a))^\sim = (1_1, 0_2)^\sim = (1_1^\sim, 0_2^\sim) = (0_1, 1_2)$.

Subsequently we be obliged to have that $g(x^\sim) = (1_2, 0_1)$.

In view of the fact that g is defined as so is f and as f is a bijection, then clearly so is g .

Therefore, g is an isomorphism.

So we categorize $x^\sim$ is a Birkhoff's central element. Hence, $x^\sim \in BC(A)$.

2.2. Lemma:

Let t be an element in a Pre A^* - algebra A . Then $tA = \{t \wedge \alpha / \alpha \in A\}$ is a Pre A^* - algebra by the induced operations $\wedge$ and $\vee$ of A and the unary operation defined by $(t \wedge \alpha)^* = t \wedge \alpha^\sim$.

Proof: Given that $tA = \{t \wedge a / a \in A\}$.

Let us choose the elements, $t \wedge x, t \wedge y, t \wedge z$ from the set tA , where x, y, z are in A .

(1) Reflect on, $(t \wedge x)^{**} = (t \wedge x^\sim)^* = t \wedge x^{\sim\sim} = t \wedge x$.

As a result, $(t \wedge x)^{**} = t \wedge x$.

(2) Regard as, $(t \wedge x) \wedge (t \wedge x) = t \wedge (x \wedge x) = t \wedge x$.

Therefore, $(t \wedge x) \wedge (t \wedge x) = t \wedge x$.

(3) Mull over, $(t \wedge x) \wedge (t \wedge y) = t \wedge (x \wedge y) = t \wedge (y \wedge x) = (t \wedge y) \wedge (t \wedge x)$.

Therefore, $(t \wedge x) \wedge (t \wedge y) = (t \wedge y) \wedge (t \wedge x)$.

(4) Consider, $((t \wedge x) \wedge (t \wedge y))^* = (t \wedge (x \wedge y))^* = t \wedge (x \wedge y)^\sim = t \wedge (x^\sim \vee y^\sim)$
 $= (t \wedge x^\sim) \vee (t \wedge y^\sim) = (t \wedge x)^* \vee (t \wedge y)^*$.

Therefore, $((t \wedge x) \wedge (t \wedge y))^* = (t \wedge x)^* \vee (t \wedge y)^*$.

(5) Consider, $(t \wedge x \wedge ((t \wedge y) \wedge (t \wedge z))) = t \wedge (x \wedge (y \wedge z)) = t \wedge ((x \wedge y) \wedge z)$
 $= ((t \wedge x) \wedge (t \wedge y)) \wedge (t \wedge z)$.

Therefore, $(t \wedge x) \wedge ((t \wedge y) \wedge (t \wedge z)) = ((t \wedge x) \wedge (t \wedge y)) \wedge (t \wedge z)$.

(6) Consider, $(t \wedge x) \wedge ((t \wedge y) \vee (t \wedge z)) = (t \wedge x) \wedge (t \wedge (y \vee z))$

$$\begin{aligned}
&= t \wedge (x \wedge (y \vee z)) = t \wedge ((x \wedge y) \vee (x \wedge z)) \\
&= (t \wedge (x \wedge y)) \vee (t \wedge (x \wedge z)) \\
&= ((t \wedge x) \wedge (t \wedge y)) \vee ((t \wedge x) \wedge (t \wedge z)).
\end{aligned}$$

Therefore, $(t \wedge x) \wedge ((t \wedge y) \vee (t \wedge z)) = ((t \wedge x) \wedge (t \wedge y)) \vee ((t \wedge x) \wedge (t \wedge z))$.

(7) Consider, $(t \wedge x) \wedge ((t \wedge x)^* \vee (t \wedge y))$

$$\begin{aligned}
&= (t \wedge x) \wedge ((t \wedge x)^{\sim} \vee (t \wedge y)) \\
&= (t \wedge x) \wedge (t \wedge (x^{\sim} \vee y)) \\
&= t \wedge (x \wedge (x^{\sim} \vee y)) \\
&= t \wedge (x \wedge y) \\
&= (t \wedge x) \wedge (t \wedge y).
\end{aligned}$$

As a result, $(t \wedge x) \wedge ((t \wedge x)^* \vee (t \wedge y)) = (t \wedge x) \wedge (t \wedge y)$.

As a consequence, $(tA, \wedge, \vee, *)$ is a Pre A^* -algebra.

2.3. Lemma: $BC(A)$ is a Pre A^* - algebra.

Proof: Let a, b be any elements from $BC(A)$. Then there exists Pre A^* - algebras A_1, A_2 and A_3, A_4 with 1 and isomorphisms $f: A \rightarrow A_1 \times A_2$ such that $f(a) = (1_1, 0_2)$ and $g: A \rightarrow A_3 \times A_4$ such that $g(b) = (1_3, 0_4)$.

Now, we have to prove that $a \wedge b$ is an element in $BC(A)$. That is, we have to find an isomorphism $h: A \rightarrow A_5 \times A_6$ such that $h(a \wedge b) = (1_5, 0_6)$ (where $1_5 \in A_5$ and $0_6 \in A_6$).

Suppose that $g(a) = (t_3, t_4)$, where $t_3 \in A_3$ and $t_4 \in A_4$.

Define, $A_5 = t_3 A_3$ where $t_3 (=1_3)$ is a meet identity in A_3 and $t_3 \wedge t_3^{\sim} (=0_3)$ is a join identity and $t_3 A_3 = \{t_3 \wedge \alpha / \alpha \in A_3\}$.

As a result of lemma 2.2, $t_3 A_3$ is a Pre A^* - algebra with $1_5 = t_3$ (meet identity in A_5) and join identity $0_5 (= t_3 \wedge t_3^{\sim} = 0_3)$.

In addition, define, $A_6 = t_4 A_4 \times A_2$. Then A_6 is also a Pre A^* - algebra in the company of meet identity $1_6 = (t_4, 1_2) (= 1_4, 1_2)$, and join identity $0_6 = (t_4 \wedge t_4^{\sim}, t_2 \wedge t_2^{\sim}) (= (0_4, 0_2))$.

Note that $0_2 = t_2 \wedge t_2^{\sim} = 0_2 \wedge t_2$.

For any x in A , let $f(x) = (s_1, s_2)$ and $g(x) = (x_3, x_4)$ where $s_1 \in A_1, s_2 \in A_2$ and $x_3 \in A_3, x_4 \in A_4$.

Define $h: A \rightarrow A_5 \times A_6$ by $h(x) = (t_3 \wedge x_3, ((t_4 \wedge x_4), s_2))$ for any $x \in A$.

Let $f(y) = (r_1, r_2)$ and $g(y) = (y_3, y_4)$.

Subsequently, $f(x \wedge y) = (s_1 \wedge r_1, s_2 \wedge r_2)$, $g(x \wedge y) = (x_3 \wedge y_3, x_4 \wedge y_4)$, $f(x^\sim) = (s_1^\sim, s_2^\sim)$ and $g(x^\sim) = (x_3^\sim, x_4^\sim)$ as f and g are isomorphisms.

$$\begin{aligned} \text{Consider, } h(x \wedge y) &= (t_3 \wedge x_3 \wedge y_3, (t_4 \wedge x_4 \wedge y_4, s_2 \wedge r_2)) \\ &= (t_3 \wedge x_3 \wedge t_3 \wedge y_3, (t_4 \wedge x_4 \wedge t_4 \wedge y_4, s_2 \wedge r_2)) \\ &= (t_3 \wedge x_3, (t_4 \wedge x_4, s_2)) \wedge (t_3 \wedge y_3, (t_4 \wedge y_4, r_2)) \\ &= h(x) \wedge h(y). \end{aligned}$$

$$\begin{aligned} \text{Now consider, } h(x^\sim) &= (t_3 \wedge x_3^\sim, (t_4 \wedge x_4^\sim, s_2^\sim)) \text{ (since } (t_3 \wedge x_3)^* = t_3 \wedge x_3^\sim) \\ &= (x_3^*, (x_4^*, s_2^\sim)) \\ &= (h(x))^\sim. \end{aligned}$$

$$\begin{aligned} \text{Consider, } h(x \vee y) &= (t_3 \wedge (x_3 \vee y_3), (t_4 \wedge (x_4 \vee y_4, s_2 \vee r_2)) \\ &= ((t_3 \wedge x_3) \vee (t_3 \wedge y_3), ((t_4 \wedge x_4) \vee (t_4 \wedge y_4), s_2 \vee r_2)) \\ &= (t_3 \wedge x_3, (t_4 \wedge x_4, s_2)) \vee (t_3 \wedge y_3, (t_4 \wedge y_4, r_2)) \\ &= h(x) \vee h(y). \end{aligned}$$

In view of that, h is a homomorphism.

To show h is injective, first we prove $h(a \wedge b) = (1_5, 0_6)$. ($\in A_5 \times A_6 = t_3 A_3 \times (t_4 A_4 \times A_2)$).

Note that $1_5 \in A_5 = t_3 A_3 = \{t_3 \wedge \alpha / \alpha \in A_3\}$ and $0_6 \in A_6 = t_4 A_4 \times A_2 = \{t_4 \wedge \alpha / \alpha \in A_4\} \times A_2$, where 1_5 is the meet identity in $t_3 A_3$ and 0_6 is the join identity in $t_4 A_4 \times A_2$.

We have, $f(a) = (1_1, 0_2)$, $g(a) = (t_3, t_4) (= (1_3, t_4))$, $g(b) = (1_3, 0_4)$, $f(b) = (t_1, t_2)$.

Now consider, $h(a \wedge b) = h(a) \wedge h(b)$ (since h is a homomorphism)

(as $h(a), h(b) \in A_5 \times A_6 = t_3 A_3 \times t_4 A_4 \times A_2$ and a, b are Birkhoff central elements of A)

$$\begin{aligned} &= (t_3 \wedge t_3, (t_4 \wedge t_4, 0_2)) \wedge (t_3 \wedge 1_3, (t_4 \wedge 0_4, t_2)) \\ &= (t_3, (t_4 \wedge 0_4, 0_2 \wedge t_2)) \\ &= (t_3, (t_4 \wedge 0_4, t_2 \wedge t_2^\sim)) \\ &= (t_3, (t_4 \wedge t_4^\sim, 0_2)) \\ &\quad \text{(by lemma 1.1(b), we have, } x \wedge 0 = x \wedge x^\sim \text{ and } t_2 \wedge t_2^\sim \text{ defined as } 0_2) \\ &= (1_5, 0_6). \end{aligned}$$

Let $x, y \in A$ such that $h(x) = h(y)$. To prove that $x = y$.

Then $t_3 \wedge x_3 = t_3 \wedge y_3$ and $t_4 \wedge x_4 = t_4 \wedge y_4$ and $s_2 = r_2$. In order to prove $x = y$ we require to prove $s_1 = r_1$ and $s_2 = r_2$. So it suffices to prove $s_1 = r_1$ as already we have $s_2 = r_2$.

Now consider $g(a) \wedge g(x) = (t_3, t_4) \wedge (x_3, x_4)$

$$= (t_3 \wedge x_3, t_4 \wedge x_4)$$

$$= (t_3 \wedge y_3, t_4 \wedge y_4)$$

$$= g(a) \wedge g(y).$$

Since g is a homomorphism, $g(a \wedge x) = g(a \wedge y)$.

This implies, $a \wedge x = a \wedge y$ (since g is one-one).

Subsequently, $f(a \wedge x) = f(a \wedge y)$ (since f is well defined).

Hence, $f(a) \wedge f(x) = f(a) \wedge f(y)$ (since f is a homomorphism).

This leads to, $(1_1, 0_2) \wedge (s_1, s_2) = (1_1, 0_2) \wedge (r_1, r_2)$.

Hence, $(1_1 \wedge s_1, 0_2 \wedge s_2) = (1_1 \wedge r_1, 0_2 \wedge r_2)$.

This implies, $(s_1, 0_2 \wedge s_2) = (r_1, 0_2 \wedge r_2)$.

Thus, $s_1 = r_1$ and $s_2 = r_2$ (already in the above we have, $s_2 = r_2$)

Therefore, $(s_1, s_2) = (r_1, r_2)$.

So, $f(x) = f(y)$.

This implies, $x = y$ (since f is one-one).

Therefore, h is one – one.

Let $(x, y) \in A_5 \times A_6$. Then $(x, y) = (t_3 \wedge x_3, (t_4 \wedge x_4, s_2))$ for some $x_3 \in A_3$, $x_4 \in A_4$ and $s_2 \in A_2$.

Since $t_3 \wedge x_3 \in t_3 A_3 \subseteq A_3$, $(t_3 \wedge x_3, t_4 \wedge x_4) \in A_3 \times A_4$ and g is onto, there exist $t \in A$ such that $g(t) = (t_3 \wedge x_3, t_4 \wedge x_4)$.

Now $g(a \wedge t) = g(a) \wedge g(t)$

$$= (t_3, t_4) \wedge (t_3 \wedge x_3, t_4 \wedge x_4)$$

$$= (t_3 \wedge t_3 \wedge x_3, t_4 \wedge t_4 \wedge x_4)$$

$$= (t_3 \wedge x_3, t_4 \wedge x_4)$$

$$= g(t)$$

Therefore, $g(a \wedge t) = g(t)$ ----- 1

This implies, $a \wedge t = t$ (since g is one-one).

Hence, $f(a \wedge t) = f(t)$ (since f is well defined).

Then we have, $f(a) \wedge f(t) = f(t)$ (since f is a homomorphism).

This leads to $(1_1, 0_2) \wedge (y_1, y_2) = (y_1, y_2)$ (since $t \in A$),

(Here, $f(t) = (y_1, y_2)$, where, $y_1 \in A_1$, $y_2 \in A_2$).

As a result, $(1_1 \wedge y_1, 0_2 \wedge y_2) = (y_1, y_2)$.

Consequently, $(y_1, 0_2 \wedge y_2) = (y_1, y_2)$. ----- 2

Therefore, $y_2 = 0_2 \wedge y_2$

Now, by above we observe that, $y_1 \in A_1, y_2 \in A_2$. Subsequently, $(y_1, y_2) \in A_1 \times A_2$.

Since f is onto, there exists, say $n \in A$ such that $f(n) = (y_1, y_2)$.

Now consider, $f(a \wedge n) = f(a) \wedge f(n)$

$$= (1_1, 0_2) \wedge (y_1, y_2)$$

$$= (1_1 \wedge y_1, 0_2 \wedge y_2)$$

$$= (y_1, 0_2 \wedge y_2)$$

$$= (y_1, y_2). \text{ (by (2))}$$

$$= f(t).$$

Since f is one-one, $a \wedge n = t$ and since g is well defined $g(a \wedge n) = g(t)$ ----- 3

Also, since, $n \in A$ we have, $g(n) = (z_1, z_2)$.

This implies, $(t_3 \wedge t_3 \wedge x_3, t_4 \wedge t_4 \wedge x_4) = g(a \wedge t)$

$$= g(t) \text{ (by (1))}$$

$$= g(a \wedge n) \text{ (by (3))}$$

$$= g(a) \wedge g(n) \text{ (since } g \text{ is a homomorphism)}$$

$$= (t_3, x_4) \wedge (z_1, z_2)$$

$$= (t_3 \wedge z_1, t_4 \wedge z_2).$$

Therefore, $t_3 \wedge t_3 \wedge x_3 = t_3 \wedge z_1$ and $t_4 \wedge t_4 \wedge x_4 = t_4 \wedge z_2$ ----- 4

Now consider, $h(n) = (t_3 \wedge z_1, t_4 \wedge z_2, s_2)$

$$= (t_3 \wedge t_3 \wedge x_3, (t_4 \wedge t_4 \wedge x_4, s_2))$$

$$= (t_3 \wedge x_3, (t_4 \wedge x_4, s_2)) \text{ (by (4))}$$

$$= (x, y).$$

Therefore, h is onto. Since, $a, b \in BC(A)$ implies $a \wedge b \in BC(A)$ and by lemma 2.1,

$a \in BC(A)$ implies $a^\sim \in BC(A)$. In addition to this, $a \vee b \in BC(A)$.

As a result, $BC(A)$ is a sub-algebra of a Pre A^* - algebra A and for this reason $BC(A)$ is a Pre A^* - algebra.

2.1. Note: Let us bring to mind the designation of centre of a Pre A^* - algebra [6]. Let $BC(A)$ be a Pre A^* - algebra with meet identity 1. Then the centre of $BC(A)$ is defined as the set $B(BC(A)) = \{a \in BC(A) / a \vee a^\sim = 1\}$.

One can see that $B(BC(A))$ is a Boolean algebra under the operations induced by those on $BC(A)$.

2.4. Lemma:

Let $a \in B(BC(A))$. Then for all x in $BC(A)$, $a \wedge x = a$ if and only if, $a \vee x = x$.

Proof: Suppose that $a \wedge x = a$.

Consider, $a \vee x$

$$= (a \wedge x) \vee x \text{ (since } a \wedge x = a \text{)}$$

$$= [a \wedge (a \sim \vee x)] \vee x \text{ (by axiom (vii) of definition [1.1], we have, } x \wedge y = x \wedge (x \sim \vee y))$$

$$= (a \vee x) \wedge [(a \sim \vee x) \vee x]$$

$$= (a \vee x) \wedge (a \sim \vee x)$$

$$= (a \wedge a \sim) \vee x = 0 \vee x = x.$$

Consequently, $a \vee x = x$.

On the other hand presume that $a \vee x = x$.

Consider, $a \wedge x = a \wedge (a \vee x) = a$ (since a in $B(BC(A))$).

Hence, $a \wedge x = a$.

2.5. Lemma:

Let $BC(A)$ be a Pre A^* - algebra and a be an element in $B(BC(A))$. In case, the set $S_a = \{ x \in BC(A) / a \wedge x = a \}$, then S_a is closed under the operations $\wedge$ and $\vee$. Also for any x in the set S_a , define, $x^* = a \vee x \sim$. Then $(S_a, \wedge, \vee, ^*)$ is a Pre A^* -algebra.

Proof: Let x, y, z be from the set S_a . Then, $a \wedge x = a$ and $a \wedge y = a$, $a \wedge z = a$.

This entails, $a \vee x = x$ and $a \vee y = y$, $a \vee z = z$. (By above lemma [2.4])

Now reflect on, $a \wedge (x \wedge y) = (a \wedge x) \wedge y = a \wedge y = a$.

Hence, $x \wedge y$ belongs to the set S_a .

Also consider, $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = a \vee a = a$.

This implies, $x \vee y$ is an element in the set S_a .

Consequently, S_a is closed under the operation $\wedge$ and $\vee$.

Reflect on, $a \wedge x^* = a \wedge (a \vee x \sim) = a$ (since a is in $B(A)$).

This involves, x^* belongs to S_a .

In consequence S_a is closed under $*$.

Now we have the following:

$$\begin{aligned} (1) \text{ Regard as, } x^{**} &= (a \vee x^{\sim})^* \\ &= a \vee (a \vee x^{\sim})^{\sim} = a \vee (x^{\sim} \wedge x) \\ &= (a \vee a^{\sim}) \wedge (a \vee x) = a \vee x \text{ (as } a \text{ is a Boolean element, } a \vee a^{\sim} = 1) = x. \end{aligned}$$

For that reason, $x^{**} = x$.

$$(2) \text{ Reflect on, } x \wedge x = (a \vee x) \wedge (a \vee x) = a \vee (x \wedge x) = a \vee x = x.$$

As a result, $x \wedge x = x$.

$$(3) \text{ By Considering, } x \wedge y = (a \vee x) \wedge (a \vee y) = (a \vee y) \wedge (a \vee x) = y \wedge x.$$

Accordingly, $x \wedge y = y \wedge x$.

$$\begin{aligned} (4) \text{ Regard as, } (x \wedge y)^* &= a \vee (x \wedge y)^{\sim} \\ &= a \vee (x^{\sim} \vee y^{\sim}) = (a \vee x^{\sim}) \vee (a \vee y^{\sim}) = x^* \vee y^*. \end{aligned}$$

Consequently, $(x \wedge y)^* = x^* \vee y^*$.

$$\begin{aligned} (5) \text{ Consider, } x \wedge (y \wedge z) &= (a \vee x) \wedge \{(a \vee y) \wedge (a \vee z)\} \\ &= a \vee \{x \wedge (y \wedge z)\} \\ &= a \vee \{(x \wedge y) \wedge z\} \text{ (since } x, y, z \text{ are in } A) \\ &= (x \wedge y) \wedge z. \end{aligned}$$

Thus, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

$$\begin{aligned} (6) \text{ Consider, } x \wedge (y \vee z) &= (a \vee x) \wedge \{(a \vee y) \vee (a \vee z)\} \\ &= \{(a \vee x) \wedge (a \vee y)\} \vee \{(a \vee x) \wedge (a \vee z)\} \\ &= \{a \vee (x \wedge y)\} \vee \{a \vee (x \wedge z)\} \\ &= (x \wedge y) \vee (x \wedge z). \end{aligned}$$

Therefore, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

$$\begin{aligned} (7) \text{ Consider, } x \wedge (x^* \vee y) &= x \wedge \{(a \vee x^{\sim}) \vee y\} \\ &= \{x \wedge (a \vee x^{\sim})\} \vee (x \wedge y) \\ &= (x \wedge x^{\sim}) \vee (x \wedge y) \text{ (since } a \vee x = x) \\ &= x \wedge (x^{\sim} \vee y) \\ &= x \wedge y. \end{aligned}$$

Therefore, $x \wedge (x^* \vee y) = x \wedge y$.

Thus, $(S_a, \wedge, \vee, *)$ is a Pre A^* -algebra.

2.6. Lemma:

Let $BC(A)$ be a Pre A^* - algebra and $a \in B(BC(A))$. Then, $f_a : BC(A) \rightarrow S_a$ is an anti-homomorphism.

Proof: Let $f_a : BC(A) \rightarrow S_a$ be a mapping defined by $f_a(x) = a \vee x^\sim$.

Consider, $f_a(x \wedge y) = a \vee (x \wedge y)^\sim$

$$= a \vee (x^\sim \vee y^\sim)$$

$$= (a \vee x^\sim) \vee (a \vee y^\sim)$$

$$= f_a(x) \vee f_a(y).$$

Therefore, $f_a(x \wedge y) = f_a(x) \vee f_a(y)$.

Again consider, $f_a(x \vee y) = a \vee (x \vee y)^\sim$

$$= (a \vee x^\sim) \wedge (a \vee y^\sim)$$

$$= f_a(x) \wedge f_a(y).$$

Therefore, $f_a(x \vee y) = f_a(x) \wedge f_a(y)$.

Finally, consider, $[f_a(x)]^* = (a \vee x^\sim)^*$

$$= a \vee (a \vee x^\sim)^\sim$$

$$= a \vee (a^\sim \wedge x)$$

$$= a \vee x$$

Similarly on the other hand consider, $f_a(x^*) = a \vee (x^*)^\sim$

$$= a \vee (a \vee x^\sim)^\sim$$

$$= a \vee (a^\sim \wedge x)$$

$$= a \vee x$$

Therefore we must have, $[f_a(x)]^* = f_a(x^*)$.

Therefore, $f_a : BC(A) \rightarrow S_a$ is an anti-homomorphism.

3. CONCLUSION

This study has been endow with the notion of Birkhoff's centre of a Pre A^* -algebra and concerned results as well. In fact, it is pragmatic that the set of all Birkhoff's central elements of a Pre A^* -algebra, that is; Birkhoff centre of Pre A^* -algebra, structures yet again a Pre A^* -algebra. Auxiliary, it is acknowledged that the set of all central elements of a Birkhoff centre of Pre A^* -algebra shapes again a Boolean algebra. It is identified that centre of the Birkhoff centre of a Pre A^* -algebra is a Boolean algebra and any element a of such algebra satisfies $a \wedge x = a$ if and only if $a \vee x = x$ for

all x in Birkhoff centre of a Pre A^* -algebra. A crucial set $S_a = \{x \in BC(A) / a \wedge x = a\}$, was defined by taking an element a from the Pre A^* -algebra (the Birkhoff's centre of a Pre A^* -algebra) and proved that it is also again a Pre A^* -algebra. Finally, it is obtained an anti-homorphism between those algebras.

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